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Automorphic Forms and Poincaré Series for Infinitely Generated Fuchsian Groups

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§1. Statement of results.

1. Let D be a simply connected domain in the extended complex plane with at least two boundary points, and G a discrete group of conformal self-mappings $z \to A(z)$ of D. If D is the upper half-plane U or the unit disc Δ the elements $A \in G$ are Möbius transformations and G is a Fuchsian group (or a Fuchsoid group in Poincaré's original terminology since we do not assume G to be finitely generated). While this can be always achieved by a conformal mapping, there are some advantages in considering the seemingly more general case of an arbitrary D.

Let

$$q \geq 2 \tag{1}$$

be a fixed integer. An automorphic form of weight (-2q) is a holomorphic solution of the functional equation

$$\phi(A(z))A'(z)^q = \phi(z) \quad \text{for} \quad z \in D\ , \quad A \in G\ . \tag{2}$$

We require in addition that

$$\phi(z) = O(|z|^{-2q})\ , \quad z \to \infty \quad \text{if} \quad \infty \in D\ . \tag{3}$$

Let $\lambda_D(z)|dz|$ denote the Poincaré metric in D. The automorphic forms with

$$\|\phi\|_{A_q(D,G)} = \iint_{D/G} \lambda_D(z)^{2-q}|\phi(z)|dxdy < \infty \tag{4}$$

form the Banach space $A_q(D,G)$ of integrable forms. The automorphic forms with

(5) $$\|\phi\|_{B_q(D,G)} = \text{ess. sup}\ \lambda_D(z)^{-q}|\phi(z)|$$

form the Banach space $B_q(D,G)$ of bounded forms. For $\phi \in A_q(D,G)$, $\psi \in B_q(D,G)$ the Petersson scalar product is defined by

(6) $$(\phi,\psi)_{q,G} = \iint_{D/G} \lambda_D(z)^{2-2q}\phi(z)\overline{\psi(z)}dxdy .$$

In (4) and (6) the integration is performed over an arbitrary fundamental region ω of G in D. This means that $\omega \subset D$ is measurable, mes Int (ω) = mes ω, $A(z_1) = z_2$ for $z_1, z_2 \in$ Int (ω) and $\mathrm{id} \neq A \in G$, and $D = \bigcup_{A\in G} A(\omega)$.

If $G = \{\mathrm{id}\}$, we write $A_q(D)$, $B_q(D)$ and $(\phi,\psi)_q$ instead of $A_q(D,G)$ $B_q(D,G)$ and $(\phi,\psi)_{q,G}$. Clearly $A_q(D,G) \cap A_q(D) = \{0\}$ unless G is finite, while $B_q(D,G)$ is always a closed linear subspace of $B_q(D)$.

2. If $D = U$ and G has a fundamental region of finite non-Euclidean area,

(7) $$\iint_{D/G} \lambda_D(z)^2 dxdy < \infty ,$$

then $A_q(D,G) = B_q(D,G)$ is the finite dimensional space of so-called cusp forms. In the general case we have

Theorem 1. The Petersson product establishes an anti-isomorphism between $B_q(D,G)$ and the dual space to $A_q(D,G)$.

It is trivial that, for a fixed $\psi \in B_q(D,G)$,

$$\ell(\phi) = (\phi,\psi)_{q,G}$$

is a continuous linear functional on $A_q(D,G)$, of norm $\|\ell\| \leq \|\psi\|_{A_q(D,G)}$. To prove Theorem 1 we will have to show that every ℓ can be so represented and that $\psi = 0$ whenever $(\phi,\psi)_{q,G} = 0$ for all $\phi \in A_q(D,G)$.

3. Let $\Phi(z)$, $z \in D$, be a holomorphic function. We say that $\Theta_{q,G}\Phi$ exists if

$$(8) \qquad (\Theta_{q,G}\Phi)(z) = \sum_{A \in G} \Phi(A(z))A'(z)^q$$

where the Poincaré series to the right converges absolutely and uniformly on compact subsets of D. In this case $\Theta_{q,G}\Phi$ is an automorphic form of weight $(-2q)$. It is known that if (7) holds, every cusp form is a Poincaré series. In the general case we have

Theorem 2. $\Theta_{q,G}$ *is a continuous mapping of* $A_q(D)$ *onto* $A_q(D,G)$.

Thus, for $\Phi \in A_q(D)$, $\Theta_{q,G}\Phi$ exists and every $\phi \in A_q(D,G)$ is of this form. If $\Phi \in B_q(D)$, however, the series in (8) may diverge. It will certainly do so if G is infinite and $\Phi \in B_q(D,G)$. Nevertheless we have

Theorem 3. *Every* $\psi \in B_q(D,G)$ *is of the form* $\psi = \Theta_{q,G}\Psi$, $\Psi \in B_q(D)$.

Theorems 2 and 3 supercede the results of [2]. For the sake of completeness we shall repeat some arguments from that paper.

4. Assume now that D = U (the upper half-plane). Following Eichler [5] we assign to every automorphic form ϕ of weight $(-2q)$ an element of the 1-dimensional cohomology group of G with coefficients in the additive group of polynomials in one variable

of degree at most 2q-2, the <u>Eichler class</u> of ϕ (cf. <u>20</u> below). It is known that under hypothesis (7) a cusp form is uniquely determined by its Eichler class.

Theorem 4. *If* $D = U$, G *is of the first kind, and the Eichler class of* $\phi \in B_q(U,G)$ *vanishes, then* $\phi = 0$.

We recall that G is said to be of the *first* or *second* kind according to whether the whole real axis is or is not contained in the closure $\Lambda(G)$ of the set of fixed points of elements of G. If G is of the second kind, $\Lambda(G)$ is either a perfect nowhere dense set or contains less than three points. In the latter case G is called *elementary*.

Theorem 5. *Let* $D = U$ *and let* G *be a non-elementary group of the second kind. The Eichler class of* $\phi \in B_q(U,G)$ *vanishes if and only if* ϕ *is orthogonal to all forms* $\Theta_{q,G}\Phi$, *where* $\Phi \in A_q(U)$ *is a rational function with poles in* $\Lambda(G)$.

If G is of the second kind, we denote by $A_2^{\#}(U,G)$ the set of those $\phi \in A_2(U,G)$ which are *continuous* and *real* on the real axis off $\Lambda(G)$.

Theorem 6. *Let* G *be as in Theorem 5. The Eichler class of* $\phi \in B_2(U,Q)$ *vanishes if and only if* ϕ *is orthogonal to* $A_2^{\#}(U,G)$.

In Theorems 5 and 6 orthogonality is meant in the sense of the Petersson product. Theorem 4 and suitably modified forms of Theorems 5 and 6 hold also for $D = \Delta$ (the unit disc).

<u>5</u>. Let D_G denote the set D from which the fixed points of elements of G distinct from the identity have been removed. The set D/G has a canonical conformal structure defined by the

requirement that the projection $D \to D/G$ be a holomorphic mapping. Thus D/G and $D_G/G \subset D/G$ are Riemann surfaces. Let π_1 denote the fundamental group.

Theorem 7. G <u>is finitely generated if and only if</u> $\pi_1(D_G/G)$ <u>is</u>.

The statement is trivial if G is a fixed point free in D, (for then $D = D_G$ and since D is simply connected G is isomorphic to $\pi_1(D/G)$). It is "well known" in all cases. But a direct proof has the advantage of enabling one to base the theory of finitely generated Fuchsian groups on uniformization theory to which an easy access via quasi-conformal mappings is now available (cf. [2]). Recently Ahlfors [1] extended Theorem 7 to Kleinian groups. Our proof of Theorem 7 is based on Theorems 4 and 6. We remark that while the proof of Theorem 4 is almost trivial the reduction of Theorem 6 to Theorem 5 depends on a device employed by Ahlfors.

§2. Preliminaries.

<u>6</u>. Let $f(z)$ be a conformal mapping of D. The Poincaré metric has the property that

(9) $\lambda_D(z)|dz|$ is a conformal invariant.

This means that $\lambda_{f(D)}(f(z))|f'(z)| = \lambda_D(z)$.

For every $A \in G$ set $\hat{A} = f \circ A \circ f^{-1}$. These $\hat{A}$'s form a discrete group $\hat{G}$ of conformal self-mappings of $f(D)$. For every function $\phi(\zeta)$, $\zeta \in f(D)$, set $(f^*\phi)(z) = \phi(f(z))f'(z)^q$. Noting condition (3) we verify that f^* is an isometric linear mapping of $A_q(f(D),\hat{G})$

onto $A_q(D,G)$ and of $B_q(f(D),\hat{G})$ onto $B_q(D,G)$ which preserves the Petersson product:

$$(f^*\phi, f^*\psi)_{q,\hat{G}} = (\phi,\psi)_{q,G} \,.$$

One also verifies that

$$\circledast_{q,G} f^*\Phi = f^* \circledast_{q,\hat{G}}\Phi$$

where the existence of one side implies that of the other. Hence it suffices to prove Theorems 1-3 for some fixed domain D.

7. We have that

$$A_q(D) \subset B_q(D) \,, \tag{10}$$

this injection being a continuous mapping.

It suffices to prove this for D = U (cf. 6) and since

$$\lambda_U(z) = |z - \bar{z}|^{-1} \tag{11}$$

the assertion follows by a standard estimate:

$$|\phi(z)| \leq \frac{4}{\pi y^2} \iint_{|\zeta - z| < y/2} |\phi(\zeta)|\, d\xi d\eta$$

$$\leq \frac{4}{\pi y^2} \iint_{|\zeta - z| < y/2} \left(\frac{2\eta}{3y}\right)^{q-2} |\phi(\zeta)|\, d\xi d\eta \leq \frac{2^q}{3^{q-2}\pi y^2} \iint_{\eta > 0} \eta^{q-2} |\phi(\zeta)|\, d\xi d\eta$$

so that $\|\phi\|_{B_q(U)} \leq 2^q 3^{2-q} \pi^{-1} \|\phi\|_{A_q(U)}$.

8. The Bergman kernel function $k_D(z,\zeta)$, $z \in D$, $\zeta \in D$ may be defined by the requirements

$$k_U(z,\zeta) = -1/\pi(z-\bar{\zeta})^2 \,, \tag{12}$$

(13) $\qquad k_D(z,\zeta)dz d\bar{\zeta}$ is a conformal invariant,

which means that $k_{f(D)}(f(z),f(\zeta))f'(z)\overline{f'(\zeta)} = k_D(z,\zeta)$ for every conformal mapping f of D. The kernel $k_D(z,\zeta)$ is a holomorphic function of z and $\bar{\zeta}$ and

(14) $$k_D(\zeta,z) = \overline{k_D(z,\zeta)}\ , \qquad \pi k_D(z,z) = \lambda_D(z)^2\ .$$

Also,

(15) $$\iint\limits_D \lambda_D(\zeta)^{2-q}|k_D(z,\zeta)|^q d\xi d\eta = C_q\lambda_D(z)^q$$

where C_q is a constant. In view of (9) and (12) it suffices to verify this for D = U in which case (15) follows from the identity

$$\iint\limits_{\eta>0} \frac{y^q\eta^{q-2}d\xi d\eta}{|x+iy-\xi+i\eta|^{2q}} = \int_{-\infty}^{+\infty}\frac{d\xi}{(1+\xi^2)^q}\int_0^{+\infty}\frac{\eta^{q-2}d\eta}{(1+\eta)^{2q-1}}$$

for y > 0.

From now on we omit the subscript D. The following *reproducing formula* holds (as it does also in bounded homogeneous domains in several variables, cf. Selberg [6]):

(16) $$\phi(z) = c_q \iint\limits_D \lambda(\zeta)^{2-2q}k(z,\zeta)^q\phi(\zeta)d\xi d\eta$$

for $\phi \in B_q(D)$, where

(16') $$c_q = (2q-1)\pi^{q-1}\ .$$

It suffices to verify this for D = Δ. Since

$$(17) \qquad k_\Delta(z,\zeta) = 1/\pi(1-z\bar{\zeta})^2 \,, \qquad \lambda_\Delta(z) = (1-|z|^2)^{-1}$$

and

$$(2q-1) \iint\limits_{|\zeta|<1} \frac{(1-|\zeta|^2)^{2q-2}\zeta^m d\xi d\eta}{(1-z\bar{\zeta})^{2q}} = \pi z^m \,, \qquad m = 0,1,\ldots$$

the assertion follows.

9. Let $L_1(D)$ and $L_\infty(D)$ denote the usual complex Banach spaces of (equivalence classes of) integrable and bounded measurable functions in D. For $\mu \in L_1(D)$ set

$$(18) \qquad (\alpha_q\mu)(z) = c_q \iint\limits_D \lambda(\zeta)^{-q}k(z,\zeta)^q\mu(\zeta)d\xi d\eta \,,$$

and for $\nu \in L_\infty(D)$ set

$$(19) \qquad (\beta_q\nu)(z) = c_q \iint\limits_D \lambda(\zeta)^{2-q}k(z,\zeta)^q\nu(\zeta)d\xi d\eta \,.$$

By (15) the mappings α_q and β_q are continuous linear mappings of $L_1(D)$ and $L_\infty(D)$ into $A_q(D)$ and $B_q(D)$, respectively. These mappings are onto, since

$$(20) \qquad \alpha_q(\lambda^{2-q}\phi) = \phi \quad \text{for} \quad \phi \in A_q(D) \,,$$

$$(21) \qquad \beta_q(\lambda^{-q}\phi) = \phi \quad \text{for} \quad \phi \in B_q(D) \,,$$

by (10) and (16). Also

$$(22) \qquad (\alpha_q\mu,\psi)_q = \iint\limits_D \mu(z)\lambda(z)^{-q}\overline{\psi(z)}dxdy \quad \text{for} \quad \psi \in B_q(D) \,,$$

and

$$(23)\qquad (\phi,\beta_q\nu)_q = \iint_D \lambda(z)^{2-q}\phi(z)\overline{\nu(z)}dxdy \quad \text{for} \quad \phi\in A_q(D) .$$

The proof involves merely substitution into the definition (6) for $G = \{id\}$, a change of order of integration, and an application of (16).

10. Let ℓ be a continuous linear functional on $A_q(D)$. By the theorems of Hahn-Banach and F. Riesz there is a $\nu\in L_\infty(D)$ such that

$$\ell(\phi) = \iint_D \lambda_D(z)^{2-q}\phi(z)\overline{\nu(z)}dxdy .$$

Hence, by (23) we have that $\ell(\phi) = (\phi,\psi)_q$ where $\psi = \beta_q\nu$. Next, let $\psi\in B_q(D)$ be such that $(\phi,\psi)_q = 0$ for all $\phi\in A_q(D)$. Noting (22) we conclude that

$$\iint_D \lambda(z)^{-q}\overline{\psi(z)}\mu(z)dxdy = 0$$

for all $\mu\in L_1(D)$. Hence $\psi\equiv 0$. Thus we have proved Theorem 1 for the case $G = \{id\}$.

§3. Poincaré series and duality.

11. We prove now that $\Theta_q = \Theta_{q,G}$ <u>is a continuous mapping of</u> $A_q(D)$ <u>into</u> $A_q(D,G)$.

Let $\Phi\in A_q(D)$ and let ω denote a fundamental region of G in D. Then

$$\iint_{\omega} \lambda(z)^{2-q}\Big| \sum_{A\in G} \Phi(A(z))A'(z)^q\Big| dxdy$$

$$\leq \sum_{A\in G} \iint_{\omega} \lambda(z)^{2-q}|\Phi(A(z))A'(z)^q| dxdy$$

$$= \sum_{A\in G} \iint_{\omega} \lambda(A(z))^{2-q}|\Phi(A(z)|\,|A'(z)|^2 dxdy$$

$$= \sum_{A\in G} \iint_{A(\omega)} \lambda(z)^{2-q}|\Phi(z)| dxdy = \|\Phi\|_{A_q(D)} .$$

This implies the absolute and uniform convergence of the series (8) in every compact subset of a fundamental region and hence on every compact subset of D, as well as the inequality

$$\| \Theta_q \Phi\|_{A_q(D,G)} \leq \|\Phi\|_{A_q(D)} .$$

(Here we used two well known facts: L_1 convergence of holomorphic functions implies normal convergence. If $D_o \subset\subset D$ there is an $\omega_o \subset\subset \omega$ and a finite sequence $\{A_1,\ldots,A_n\} \subset G$ such that $D_o \subset A_1(\omega_o) \cup \ldots \cup A_n(\omega_o)$.)

12. Let ℓ be a continuous linear functional on $A_q(D,G)$. Let ω be a fundamental region. Then, by Hahn-Banach and F. Riesz,

$$\ell(\phi) = \iint_{\omega} \lambda(z)^{2-q}\phi(z)\nu(z) dxdy \tag{24}$$

with a bounded measurable $\nu(z)$. We extend ν over the whole of D by the relation

$$\nu(A(z))(\overline{A'(z)}/A'(z))^{q/2} = \nu(z) \quad \text{for} \quad A \in G \tag{25}$$

(where $(\bar{A}'/A')^{q/2} = |A'|^q(A')^{-q}$). For $\Phi \in A_q(D)$ we have

(26) $$\ell(\Theta_q \Phi) = \iint_D \lambda(z)^{2-q} \Phi(z) \nu(z) dxdy$$

as follows from the identiy

$$\iint_\omega \lambda(z)^{2-q} \sum \Phi(A(z)) A'(z)^q \nu(z) dxdy$$

$$= \sum_{A \in G} \iint_{A(\omega)} \lambda(z)^{2-q} \Phi(z) \nu(z) dxdy \quad .$$

Using this we shall show that

(27) $$\ell(\Theta_q \Phi) = 0 \quad \text{for all} \quad \Phi \in A_q(D)$$

implies that

(28) $$\ell(\phi) = 0 \quad \text{for all} \quad \phi \in A_q(D,G) \ ,$$

which means that

(29) $$\Theta_q A_q(D) \text{ is dense in } A_q(D,G) \ .$$

During this proof we assume that $D = \Delta$ (the unit disc) and 0 is not a fixed point of any element of G distinct from the identity. This assumption involves no loss of generality.

13. For ν satisfying (25) for $D = \Delta$ and such that the corresponding functional ℓ vanishes on $\Theta_q A_q(\Delta)$ set

(30) $$h(z) = -\frac{1}{\pi} \iint_{|\zeta|<1} \frac{(1-|\zeta|^2)^{q-2} \nu(\zeta) d\xi d\eta}{\zeta - z}$$

and, for some fixed θ, $0 < \theta < 2\pi$,

$$(31) \qquad \tilde{h}(z) = -\frac{(1-ze^{-i\theta})^{q-2}}{\pi} \iint\limits_{|\zeta|<1} \frac{(1-|\zeta|^2)^{q-2}\nu(\zeta)d\xi d\eta}{(1-\zeta e^{-i\theta})^{q-2}(\zeta-z)} .$$

For a fixed z such that $|z| \geq 1$ the functions

$$\Omega(\zeta) = -\frac{1}{\pi}\frac{1}{\zeta - z} , \qquad \tilde{\Omega}(\zeta) = -\frac{1}{\pi}\frac{1}{(1-\zeta e^{-i\theta})^{q-2}(\zeta - z)}$$

belong to $A_q(\Delta)$, and by (26)

$$h(z) = \ell(\Theta_q \Omega) , \qquad \tilde{h}(z) = (1-ze^{-i\theta})^{q-2}\ell(\Theta_q \tilde{\Omega}) ,$$

so that by (27)

$$(32) \qquad h(z) = \tilde{h}(z) = 0 \quad \text{for} \quad |z| \geq 1 .$$

From well known properites of logarithmic potentials we conclude that h and $\tilde{h}$ are continuous everywhere and that, in view of the second equation (32),

$$(33) \qquad \left| \iint\limits_{|\zeta|<1} \frac{(1-|\zeta|^2)^{q-2}\nu(\zeta)d\xi d\eta}{(1-\zeta e^{-i\theta})^{q-2}(\zeta - z)} \right| \leq c\,(1-|z|)\log\frac{1}{1-|z|}$$

for $|z| < 1$, where c does not depend on θ. Also

$$(34) \qquad \frac{\partial h}{\partial \bar{z}} = \frac{\partial \tilde{h}}{\partial \bar{z}} = (1-|z|^2)^{q-2}\nu(z) \quad \text{for} \quad |z| < 1$$

(in the sense of weak derivatives). By (32) and (34) we have that $h \equiv \tilde{h}$. Noting (33) and the fact that θ was aribtrary we conclude that

$$(35) \qquad h(z) = O(-(1-|z|)^{q-1}\log(1-|z|)) , \quad |z| \uparrow 1 .$$

One computes easily from (34) and (25) that for every fixed $A \in G$ the function

$$h(A(z))A'(z)^{1-q} - h(z)$$

is holomorphic in $|z| < 1$. Since it vanishes on $|z| = 1$ we have that

(36) $$h(A(z)) = h(z)A'(z)^{q-1} \quad \text{for} \quad A \in G .$$

Using these properties of h we shall show that $\ell \equiv 0$.

14. Let ω be the closure in Δ of the set

$$\{z \in \Delta \mid |A(z)| > |z| \quad \text{for} \quad \mathrm{id} \neq A \in G\}$$

and let ω_r be the intersection of ω with $|z| < r < 1$. Then ω is a fundamental region. For every r, $0 < r < 1$, the boundary σ_r of ω_r consists of a portion γ_r of the circle $|z| = r$ and of $2n = 2n(r)$ circular arcs $\delta_1, \ldots, \delta_n$, $\delta_1', \ldots, \delta_n'$ such that there exist elements $A_1, \ldots, A_n$ of G with

(37) $$A_j(\delta_j) = -\delta_j' , \qquad j = 1, \ldots, n .$$

All this is known and easy to check.

Now let $\phi \in A_q(\Delta, G)$ be given. By (24) and (34)

$$\ell(\phi) = \lim_{r \uparrow 1} \iint_{\omega_r} (1 - |z|^2)^{q-2} \nu(z)\phi(z)\,dxdy$$

$$= \lim_{r \uparrow 1} \iint_{\omega_r} \phi \frac{\partial h}{\partial \bar{z}}\,dxdy = \frac{i}{2} \lim_{r \uparrow 1} \int_{\sigma_r} \phi\, h\, dz .$$

Since by (34) we have that

$$\phi(A(z))h(A(z))A'(z) = \phi(z)h(z) \quad \text{for} \quad A \in G ,$$

it follows from (37) that

$$\int_{\delta_j} \phi h \, dz + \int_{\delta_j'} \phi h \, dz = 0 , \qquad j = 1,\dots,n ,$$

so that

$$-2i\ell(\phi) = \lim_{r\uparrow 1} \int_{\gamma_r} \phi h \, dz ,$$

and by (35)

$$(38) \qquad |\ell(\phi)| \leq \text{const.} \liminf_{r\uparrow 1} (1-r) \log \frac{1}{1-r} \int_{\gamma_r} |\phi| |dz| .$$

Since

$$\int_{1/2}^{1} (1-r^2) \int_{\gamma_r} |\phi| |dz| dr \leq \|\phi\|_{A_q(\Delta,G)}$$

(38) implies that $\ell(\phi) = 0$. Q.E.D.

15. For $\Phi \in A_q(D)$, $\psi \in B_q(D,G)$ we have that

$$(39) \qquad (\Phi,\psi)_q = (\Theta_q\Phi,\psi)_{q,G} .$$

Indeed this means that

$$\sum_{A\in G} \iint_{A(\omega)} \lambda(z)^{2-2q}\Phi(z)\overline{\psi(z)}dxdy$$

$$= \iint_{\omega} \lambda(z)^{2-2q}\overline{\psi(z)} \sum_{A\in G} \Phi(A(z))A'(z)^q dxdy$$

which is easily verified.

16. Proof of Theorem 1. Assume that $\psi \in B_q(D,G)$ is such that $(\phi,\psi)_{q,G} = 0$ for all $\phi \in A_q(D,G)$, then $(\Theta_q\Phi,\psi)_{q,G} = 0$ for all $\Phi \in A_q(D)$ and by (39) also $(\Phi,\psi)_q = 0$. Hence $\psi = 0$ by the result in 10.

Now let $\ell(\phi)$ be a given linear functional on $A_q(D,G)$. Then (cf. 12) there is a $\nu \in L_\infty(D)$ satisfying (25) such that (24) holds.

Set $\psi = \beta_q \overline{\nu}$. Then (cf. 9) $\psi \in B_q(D)$ and by (26) and (23)

(40) $$\ell(\circledast_q \Phi) = (\Phi, \psi)_q \quad \text{for} \quad \Phi \in A_q .$$

Now, for $A \in G$ and $B = A^{-1}$

$$\psi(A(z))A'(z)^q = c_q \iint_D A'(z)^q \lambda(\zeta)^{-q} k(z,\zeta)^q \overline{\nu(\zeta)} d\xi d\eta .$$

Setting $\zeta = B\hat{\zeta}$ and noting (9), (13) we obtain

$$\psi(A(z))A'(z)^q = c_q \iint_D A'(z)^q \lambda(\zeta)^{2-q} k(A(z),\zeta)^q \overline{\nu(\zeta)} d\xi d\eta$$

$$= c_q \iint_D \lambda(A \circ B(\zeta))^{2-q} A'(z)^q k(A(z), A \circ B(\zeta))^q \overline{\nu(A \circ B(\zeta))} d\xi d\eta$$

$$= c_q \iint_D \lambda(B(\zeta))^{2-q} k(z, B(\zeta))^q \overline{\nu(B(\zeta))} |B'(\zeta)|^2 d\xi d\eta = \psi(z) .$$

Thus $\psi \in B_q(D,G)$ and, by (39) and (40),

$$\ell(\phi) = (\phi, \psi)_{q,G}$$

whenever $\phi \in \circledast_q A_q(D)$. In view of (29) the same holds for all $\phi \in A_q(D,G)$.

17. Proof of Theorem 2. In view of 11 we must show only that $\circledast_q A_q(D) = A_q(D,G)$. Let χ be the characteristic function of a fundamental region ω. Then $\chi \lambda^{2-q} \phi \in L_\infty(D)$ and we may form $\hat{\phi} = \circledast_q \alpha_q(\chi \lambda^{2-q} \phi)$ which belongs to $A_q(D,G)$. Let ψ be any element in $B_q(D,G)$. By (39)

$$(\hat{\phi},\psi)_{q,G} = (\alpha_q(\chi\lambda^{2-q}\phi),\psi)_q$$

$$= c_q \iint_D \lambda(z)^{2-2q}\overline{\psi(z)} \iint_\omega \lambda(\zeta)^{2-2q}\phi(\zeta)k(z,\zeta)^q d\xi d\eta dx dy$$

$$= c_q \iint_\omega \lambda(\zeta)^{2-2q}\phi(\zeta) \iint_D \lambda(z)^{2-2q}\overline{k(\zeta,z)}^q\overline{\psi(z)}dx dy d\xi d\eta$$

$$= (\phi,\psi)_{q,G} .$$

Hence

(41) $$\phi = \circledast_q\alpha_q(\chi\lambda^{2-q}\phi) ,$$

by Theorem 1.

18. Proof of Theorem 3. Let χ be as in the previous proof. We shall show that if $\phi \in B_q(D,G)$, then

(42) $$\phi = \circledast_q\beta_q(\chi\lambda^{-q}\phi)$$

(note that $\chi\lambda^{-q}\phi \in L_\infty(D)$). By (16)

$$\phi(z) = \sum_{A\in G} c_q \iint_{A(\omega)} \lambda(\zeta)^{2-2q}k(z,\zeta)^q\phi(\zeta)d\xi d\eta$$

this series being absolutely and normally convergent. Setting $B = A^{-1}$ and using (2), (9) and (12) we obtain

$$\phi(z) = \sum_{A\in G} c_q \iint_{A(\omega)} \lambda(B(\zeta))^{2-2q}k(B(z),B(\zeta))^q\phi(B(\zeta))B'(z)^q|B'(\zeta)|^2 d\xi d\eta$$

$$= \sum_{A\in G} c_q B'(z)^q \iint_\omega \lambda(\zeta)^{2-2q}k(B(z),\zeta)^q\phi(\zeta)d\xi d\eta$$

which is precisely (42).

§4. Periods of automorphic forms

19. Let $D = U$ so that G is a group of Möbius transformations $z \longrightarrow A(z) = (az+b)/(cz+d)$. Let Π_{2q-2} denote the additive groups of polynomials $P(z) = \sum_{j=0}^{2q-2} \alpha_j z^j$. The group G operates from the right on Π_{2q-2} by the rule

$$(PA)(z) = P(A(z))A'(z)^{1-q} . \tag{43}$$

A mapping $A \longrightarrow P_A$ of G into Π_{2q-2} is called a cocycle of

$$P_{AB} = P_A B + P_B , \tag{44}$$

a coboundary if there exists an element $Q \in \Pi_{2q-2}$ such that

$$P_A = QA - Q . \tag{45}$$

The coboundaries form a subgroup of the additive group of cocycles. The factor group (cocycles/coboundaries) is denoted by $H^1(G, \Pi_{2q-2})$.

20. Let ϕ be an automorphic form of weight $(-2q)$ and F a holomorphic function such that

$$\frac{d^{2q-1}F(z)}{dz^{2q-1}} = \phi(z) . \tag{46}$$

One verifies easily that for every $A \in G$ the $(2q-1)$-st derivative of

$$F(A(z))A'(z)^{1-q} - F(z) \tag{47}$$

vanishes, so that this function belongs to Π_{2q-2}. We call it the Eichler period of F on A. The mapping

$$(48) \qquad A \longrightarrow F(A(z))A'(z)^{1-q} - F(z)$$

is clearly a cocycle. Since F is determined by ϕ modulo a polynomial of degree at most $2q-2$, the cohomology class of (48) depends only on ϕ and depends on ϕ linearly. We call it the Eichler class of ϕ.

The existence of an F satisfying (46) and the condition

$$(49) \qquad F(A(z))A'(z)^{1-q} = F(z) \quad \text{for all} \quad A \in G$$

is necessary and sufficient for the vanishing of the Eichler class of ϕ.

21. Let $\phi \in B_q(U,G)$. Then $|\phi(x+iy)| \leq \|\phi\|_{B_q(U,G)} y^{-q}$ so that every $F(z)$ satisfying (46) is continuous on the real axis. Assume that (49) holds and let $x \in \mathbb{R}$ be a fixed point of a hyperbolic parabola element A of G. Then $A(x) = x$, $A'(x) \neq 1$ and, by (49), $F(x) = 0$. Hence also

$$(50) \qquad F(x) = 0 \quad \text{for} \quad x \in \Lambda(G)$$

where $\Lambda(G)$ is the closure of the set of fixed points. Conversely, if (46) and (50) hold, then for every fixed $A \in G$ the polynomial (47) vanishes on $\Lambda(G)$ since $A(\Lambda(G)) = \Lambda(G)$ for every $A \in G$. If G is not elementary, $\Lambda(G)$ is infinite and we conclude that (49) holds.

22. Proof of Theorem 4. If G is of the first kind and the Eichler class of $\phi \in B_q(U,G)$ is zero, then $\phi(z) = F^{(2q-1)}(z)$ with $F = 0$ on $\mathbb{R}$. Hence $F \equiv 0$, $\phi \equiv 0$.

23. Let Λ be a perfect set on the real axis (in the next paragraph we shall take $\Lambda = \Lambda(G)$ for a non-elementary group G of

the second kind). Let $a_1,\dots,a_q$ be distinct points of Λ and set

(51) $$p(z) = (z-a_1)(z-a_2)\dots(z-a_q) .$$

Then every rational function with simple poles in Λ which belongs to $A_q(U)$ is of the form

(52) $$\sum_{j=1}^{n} \frac{\alpha_j}{(z-x_j)p(z)}$$

where $x_1,\dots,x_n$ are distinct points of Λ and $x_j \neq a_k$ and the α_j are arbitrary complex constants. Indeed, a rational function with no singularities except perhaps for simple poles at $a_1,\dots,a_q$, $x_1,\dots,x_n$ belongs to $A_q(U)$ if and only if it is of the form

$$\sum_{j=1}^{q} \frac{\beta_j}{z-a_j} + \sum_{j=1}^{n} \frac{\gamma_j}{z-x_j}$$

with

$$\sum_{j=1}^{q} \beta_j a_j^s + \sum_{j=1}^{n} \gamma_j x_j^s = 0 , \qquad s = 0,1,\dots,q-1 .$$

The space of such functions has therefore dimension n. On the other hand (52) always belongs to $A_q(U)$.

If $\Phi(z) \in A_q(U)$ is a rational function with poles in Λ it is a limit of functions of the form (52). Indeed, if $\xi_1,\dots,\xi_m$ are the poles of Φ and $\nu_1,\dots,\nu_m$ their multiplicities we have $0 < \nu_j \leq q-1$ and

$$\Phi(z) = r(z) \prod_{j=1}^{m} (z-\xi_j)^{-\nu_j}$$

where $r(z)$ is a polynomial of degree at most $\nu_1 + \dots + \nu_m + q-1$ with $r(\xi_j) \neq 0$. Let $\varepsilon > 0$ be given. Since Λ is perfect there exist

distinct points ξ_{jk}, $j = 1,\dots,m$, $k = 1,\dots,\nu_j$ in Λ with $|\xi_{jk} - \xi_j| < \varepsilon$. The function

$$\tilde{\Phi}(z) = r(z) \prod_{j=1}^{m} \prod_{k=1}^{\nu_j} (z - \xi_{jk})^{-1}$$

is of the form (52) and one verifies that $\|\Phi - \tilde{\Phi}\|_{A_q(U)}$ will be arbitrarily small for ε sufficiently small.

24. Proof of Theorem 5. Let $\Lambda = \Lambda(G)$ and let $a_1,\dots,a_q$ and $p(z)$ be as in 23. Let $\phi \in B_q(U,G)$. Noting (11), (12) we write (16) in the form

$$\phi(z) = \frac{(-1)^q(2q-1)}{\pi} \iint_{\eta>0} \frac{|\zeta - \bar{\zeta}|^{2q-2}\phi(\zeta)\,d\xi d\eta}{(\bar{\zeta} - z)^{2q}} .$$

Set

$$G(z) = \iint_{\eta>0} \frac{|\zeta - \bar{\zeta}|^{2q-2}\phi(\zeta)\,d\xi d\eta}{(\bar{\zeta} - z)p(\bar{\zeta})} .$$

This function is holomorphic in U and continuous everywhere except perhaps at the points a_j. Next, set

$$F(z) = \frac{(-1)^q p(z)G(z)}{\pi(2q-2)!} .$$

Then $F(a_j) = 0$, $j = 1,\dots,q$ and since

$$\frac{p(z)}{p(\bar{\zeta})(\bar{\zeta} - z)} - \frac{1}{\bar{\zeta} - z}$$

is a polynomial of degree q-1 in z, we have that

$$F^{(2q-1)}(z) = \frac{(-1)^q(2q-1)}{\pi} \iint_{\eta>0} \frac{|\zeta - \bar{\zeta}|^{2q-2}\phi(\zeta)\,d\xi d\eta}{(\bar{\zeta} - z)^{2q}}$$

in U, so that (46) holds. By 21 the Eichler class of ϕ vanishes if and only if $F(x) = 0$ for $x \in \Lambda(G)$, $x \neq a_j$. This condition is equivalent to

$$G(x) = 0 \quad \text{on} \quad \Lambda(G) - \{a_1, \ldots, a_q\} .$$

But for a real $x \neq a_j$

$$\pi \overline{G(x)} = (-1)^q (2q-1)(\Phi, \phi)_q$$

where

$$\Phi(x) = \frac{1}{(z-x)p(z)} \in A_q(U) .$$

The conclusion of Theorem 5 now follows from 23.

25. Let G be again a Fuchsian non-elementary group of the second kind and let Ω denote the complement of $\Lambda(G)$ in the extended complex plane. Then there exists a Fuchsian group H_o without elliptic elements and a holomorphic mapping $\zeta \rightarrow g(\zeta)$ of U onto Ω such that if $\zeta_1, \zeta_2 \in U$, then $g(\zeta_1) = g(\zeta_2)$ if and only if there is a $C \in H_o$ with $C(\zeta_1) = \zeta_2$. Also, there is a Fuchsian group H such that if $\zeta_1, \zeta_2 \in U$, then $A(g(\zeta_1)) = g(\zeta_2)$ for some $A \in G$ if and only if there is a $B \in H$ with $B(\zeta_1) = \zeta_2$. The mapping τ of H onto G which sends $B \in H$ into $A \in G$ with $g \circ B = A \circ g$ is a holomorphism; its kernel is precisely H_o.

Let $\phi \in A_2^{\prime}(U,G)$. This means that $\phi \in A_2(U,G)$ and $\phi(z)$ is holomorphic in Ω and satisfies the relation

$$\phi(\overline{z}) = \overline{\phi(z)} . \tag{53}$$

Let ω be a fundamental region for G in Ω chosen so that $\omega \cap U$ is simply connected and ω is invariant under the mapping $z \rightarrow \overline{z}$.

Then there is a fundamental region $\hat{\omega}$ for H in U such that $g(\hat{\omega}) = \omega$. Let $K \subset H$ contain exactly one representative of each coset of H modulo H_o. Then

$$\hat{\omega}_o = \bigcup_{B \in K} B(\hat{\omega})$$

is a fundamental region for H_o in U and $g(\hat{\omega}_o) = \Omega$.

Set $\hat{\phi}(\zeta) = \phi(g(\zeta))g'(\zeta)^2$. Then

$$\iint_{\omega} |\hat{\phi}(\zeta)|d\xi d\eta = \iint_{\omega} |\phi(z)|dxdy = 2\|\phi\|_{A_2(U,G)}$$

by (53), and for $B \in H$ we have that

$$\hat{\phi}(B(\zeta))B'(\zeta)^2 = \phi(g(B(\zeta))g'(B(\zeta))^2 B'(\zeta)^2$$

$$= \phi(A(g(\zeta))A'(g(\zeta))^2 g'(\zeta)^2 = \phi(g(\zeta))g'(\zeta)^2 = \hat{\phi}(\zeta)$$

where A is the image of B under the homomorphism τ described above. Hence $\hat{\phi} \in A_2(U,H)$ and by Theorem 2 we have that $\hat{\phi} = \Theta_{2,H}\hat{\Phi}$, $\hat{\Phi} \in A_2(U)$, or

(54) $$\hat{\phi}(\zeta) = \sum \hat{\Phi}(B(\zeta))B'(\zeta)^2$$

$$= \sum_{B \in K} \sum_{C \in H_o} \hat{\Phi}(C(B(\zeta))C'(B(\zeta))^2 B'(\zeta)^2 .$$

Set

$$\hat{\Phi}_o(\zeta) = \sum_{C \in H_o} \hat{\Phi}(C(\zeta))C'(\zeta)^2 .$$

Then $\hat{\Phi}_o = \Theta_{2,H_o}\hat{\Phi} \in A_2(U,H_o)$. Hence there exists a holomorphic function $\Phi_o(z)$, $z \in \Omega$ such that $\hat{\Phi}_o(\zeta) = \Phi_o(g(\zeta))g'(\zeta)^2$; we have that

$$\iint_{\Omega} |\Phi_o(z)| dxdy < \infty$$

since this integral equals $\|\hat{\Phi}_o\|_{A_2(U,H_o)}$.

Now (54) may be written as

$$\phi(g(\zeta))g'(\zeta)^2 = \sum_{B\in K} \hat{\Phi}_o(B(\zeta))B'(\zeta)^2$$

$$= \sum_{B\in K} \Phi_o(g(B(\zeta)))g'(B(\zeta))^2 B'(\zeta)^2$$

$$= \sum_{A\in G} \Phi_o(A(g(\zeta)))A'(g(\zeta))^2 g'(\zeta)^2$$

Thus _every_ $\phi \in A_2^{\#}(U,G)$ _admits the representation_

(55) $$\phi = \Theta_{2,G}\Phi_o$$

where Φ_o _is holomorphic in_ Ω _and absolutely integrable over this domain_.

_26_. _Proof of Theorem 6_. Assume that $\psi \in B_2(U,G)$ is orthogonal to $A_2^{\#}(U,G)$. Let $r(z)$ be a rational function with poles in $\Lambda(G)$ belonging to $A_2(U)$ and $\phi = \Theta_{2,G}r$. Since

$$\iint_{\Omega} |r(z)| dxdy < \infty$$

(Ω having the same meaning as in _25_) the argument in _11_ can be repeated to show that the Poincaré series

$$\sum_{A\in G} r(A(z))A'(z)^2$$

converges absolutely and normally in Ω. This implies that $\phi(z) = \phi_1(z) + i\phi_2(z)$, with $\phi_1, \phi_2 \in A_2^{\#}(U,G)$. Hence $(\phi,\psi)_{2,G} = 0$. By Theorem 5 the Eichler class of ψ is zero.

Assume next that the Eichler class of $\psi \in B_2(U,G)$ vanishes and let $\phi \in A_2^{\#}(U,G)$. Then ϕ admits the representation (55). By the approximation theorem proved in [2] there exists a sequence of rational functions $\{r_j(z)\}$ with poles in $\Lambda(G)$ such that

(56) $$\iint_{\Omega} |r_j(z) - \Phi_0(z)|dxdy \longrightarrow 0 .$$

By Theorem 5 we have that $(\Theta_{2,G}r_j, \psi)_{2,G} = 0$. Since (56) implies that $\|r_j - \Phi_0\|_{A_2(U)} \longrightarrow 0$, we have that $\Theta_{2,G}r_j \longrightarrow \phi$ in $A_2(U,G)$, by Theorem 2. Therefore $(\phi, \psi)_{2,G} = 0$.

§5. Finitely generated Fuchsian groups

27. A Riemann surface S will be called of finite type, more precisely of type (g,n,m), if it is conformally equivalent to $S_0 - \sigma$ where S_0 is a closed (compact) surface of genus g and σ a closed set with $n+m \geq 0$ components of which $n \geq 0$ are points and $m \geq 0$ simply connected non-degenerate continua. The numbers g, n, m depend only on S; we say that S has n punctures and m boundary curves.

If $m = 0$ then S_0 (the natural compactification of S) is determined by S except for conformal equivalence. If $m > 0$ there exists a Riemann surface S_1 of type $(2g+m-1, 2n)$ (the double of S) which is determined by S except for conformal equivalence, m disjoint simple closed analytic curves $\gamma_1, \ldots, \gamma_m$ on S_1 and an anticonformal involution ρ of S_1 which leaves a point $p \in S_1$ fixed if and only if $p \in \gamma = \gamma_1 \cup \ldots \cup \gamma_m$, such that $S_1 - \gamma$ consists of two components one of which is conformally equivalent to S.

The fundamental group $\pi_1(S)$ is finitely generated if and only if S is of finite type. This is a known result in surface topology.

28. Let D_G/G be of finite type. Then G is finitely generated.

This is well known and can be proved by disecting D_G/G by finitely many smooth curves into a simply connected region such that a component of its inverse image under the projection $D_G \to D_G/G$ is a fundamental domain whose boundary consists of finitely many "sides".

29. Let S be a Riemann surface. An Abelian differential (of the first kind) on S is a rule associating with every local $p \to t(p)$ defined on a domain $G \subset S$ a holomorphic function $\phi(t)$ such that $\phi(t)dt$ is invariant under parameter changes. In this case $|\phi(t)|^2$ is a density. If we demand instead the invariance of $\phi(t)dt^2$ we obtain a quadratic (holomorphic) differential; now $|\phi(t)|$ is a density. The Abelian differentials α with

$$\iint_S |\alpha|^2 < \infty$$

form a Hilbert space $A_1(S)$ of square integrable differentials. The quadratic differentials β with

$$\iint_S |\beta| < \infty \tag{57}$$

form the Banach space $A_2(S)$ of integrable differentials. We have that

$$\dim A_1(S) \leq \dim A_2(S)$$

because if $\alpha_1, \alpha_2 \in A_1(S)$, then $\alpha_1\alpha_2 \in A_2(S)$. If the genus of S is infinite, then $\dim A_1(S) = \infty$ (cf. Nevanlinna [4]) and hence $\dim A_2(S) = \infty$. If the genus of S is $g < \infty$, then $S = S_0 - \sigma$ where σ is a closed set on the closed surface S_0 of genus g. If S contains N distinct points, then $\dim A_2(S) \geq N$ since it is known (say from the Riemann-Roch theorem) that to every $p \in S_0$ there is a meromorphic quadratic differential β_p on S_0 whose only singularity is a simple pole at p. We conclude that

(58) $\quad \dim A_2(S) = \infty$ unless S is of finite type $(g,n,0)$.

<u>30</u>. The space $A_2(D,G)$ can be defined even when G is a discrete group of conformal self-mappings of a non-simply connected domain (since λ does not enter in the definition of this space). Let D_G denote D with the fixed points of elements of G (distinct from the identity) removed. Then there is a canonical isomorphism

$$A_2(D,G) \cong A_2(D_G/G) . \tag{59}$$

Indeed, $A_2(D,G)$ may be identified with the space X of meromorphic quadratic differentials β on the Riemann surface D/G for which (57) holds and which have no singularities except perhaps simple poles on the set σ consisting of the images of fixed points of G under the projection $D \to D/G$. Since σ is discrete and $D/G - \sigma = D_G/G$, X may be identified with $A_2(D_G/G)$.

<u>31</u>. Let G be a Fuchsian group. The elements of $B_q(U,G)$ with vanishing Eichler class form a closed linear subspace $B_q^0(U,G)$.

<u>If</u> G <u>is finitely generated</u>, $\dim B_q(U,G)/B_q^0(U,G) < \infty$.

[illegible]

(5.8) [illegible]

21. The [illegible] group of [illegible] symmetric [illegible] (cyclic group of [illegible] elements [illegible] of [illegible] the [illegible] of [illegible] order). [illegible] D_n [illegible] classes of G (without [illegible] identity) removed. [illegible] canonical isomorphism

$$A_k(G,Q) \cong A_k(G,\Gamma) \tag{5.9}$$

Indeed, $A_k(G,Q)$ may be identified with the space of [illegible] quadratic differentials [illegible] on the Riemann surface [illegible] (??) holds and [illegible] perhaps simple poles on the set [illegible] consisting of [illegible] points of S under the projection [illegible] and [illegible] $= S_0/G$, Z may be identified with $A_k(G,\Gamma)$.

22. [illegible] Fuchsian group. The element [illegible] with vanishing [illegible] linear [illegible] Σ is a [illegible] [illegible].

Indeed, assign to every $\phi \in B_q(U,G)$ a holomorphic function $F(z)$, $z \in U$ such that $F^{(2q-1)}(z) = \phi(z)$ and $F^{(\nu)}(z) = 0$, $\nu = 0,1,\ldots,2q-2$. Then $\phi \in B_q^0(U,G)$ whenever the Eichler periods of F vanish on a set of generators of G. This amounts to finitely many linear conditions.

32. Proof of Theorem 7. We may assume that $D = U$. We may assume that G is non-elementary, the theorem being trivial for elementary groups. In view of 27, 28 it suffices to assume that G is finitely generated and to prove that U_G/G is of finite type.

Let G be of the first kind. Then $B_2^0(U,G) = \{0\}$ by Theorem 4, hence $\dim B_2(U,G) < \infty$ by 31, hence $\dim A_2(U,G) < \infty$ by Theorem 1, hence $\dim A_2(U_G/G) < \infty$ by (59), hence U_G/G is of the finite type $(g,n,0)$ by (58).

Assume next that G is of the second kind. Let $A_2^\flat(U,G)$ denote the subspace of $A_2(U,G)$ consisting of elements of the form $\phi_1 + i\phi_2$ with ϕ_1, ϕ_2 $A_2^\#(U,G)$. By Theorems 1 and 6 the dual space to $A_2^\flat(U,G)$ is anti-isomorphic to $B_2(U,G)/B_2^0(U,G)$. Thus $\dim A_2^\flat(U,G) < \infty$. Let Ω have the same meaning as in 25. One sees at once that $A_2^\flat(U,G)$ may be identified with $A_2(\Omega,G)$. Hence $\dim A_2(\Omega_G/G) < \infty$ by (59), and in view of (58) the Riemann surface $S_1 = \Omega_G/G$ is of finite type $(g,n,0)$. The mapping $z \longrightarrow \bar{z}$ induces an anti-conformal involution ρ on S_1. The set γ of fixed points of ρ is the image of the intersection of Ω_G with the extended real axis under the canonical mapping $\Omega_G \longrightarrow S_1$ and one of the two components of $S_1 - \gamma$ is U_G/G. Hence U_G/G is of finite type (g,n,m) with $n > 0$.

Then, each [illegible] [illegible] such that [illegible]

$j = 0,1,\dots,2n-2$. [illegible]

of E vanish [illegible]. This accounts [illegible]

many [illegible] conditions.

2. [illegible]. We now assume that [illegible]. Assume that G is [illegible], the theorem being [illegible] completely proved. [illegible] G is [illegible] [illegible]

Let G be of the first kind. Then [illegible]

[illegible]

type [illegible].

Assume next that G is of the second kind and [illegible] the presence of [illegible] [illegible] Then [illegible]

[illegible]

[illegible] Hence [illegible]

[illegible].

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